Corner Rectangle Visibility Graphs

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Abstract

Corner rectangle visibility graphs (CRVGs) combine two existing lines of inquiry in graph theory: rectangle visibility graphs (RVGs) and rectangle-of-influence graphs (RIGs). An RVG uses vertical and horizontal bands of sight between axis-parallel rectangles in the plane to construct a graph whose vertices and edges correspond to rectangles and visibility bands respectively. RIGs are a straight-line embedding of a graph, where edges can be represented as empty axis-parallel rectangles of influence with adjacent vertices at opposing corners of the RI. We define CRVGs by giving each rectangle a single eye in its corner and defining visibility relations accordingly. We prove that families of graphs, including paths, cycles, wheels, trees, k-partite graphs up to $k = 4$, and complete graphs K_n for $n \leq 5$ are representable by corner rectangle diagrams. Our work further analyzes the maximum number of edges e that can be drawn in restricted CRVG representations: (1) where all rectangles look the same direction (SCRVGs), $e = \left[\frac{n^2}{4}\right] + n - 2$; and (2) where all rectangles look in orthogonal directions (SWCRVGs), $e = \left[\frac{n^2}{3} + \frac{n}{3}\right] - 1$.

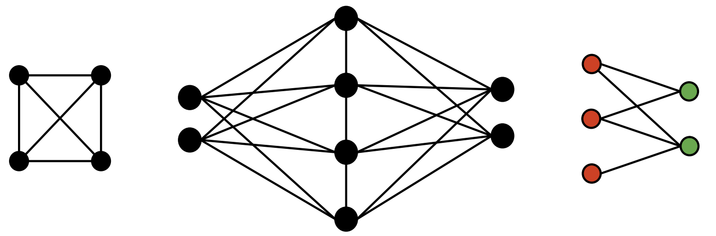


Figure 1: A complete graph (K_4), a double fan, and a bipartite graph.

1 Background

1.1 Graphs

A graph $G = (V, E)$ is a set of vertices and edges (or alternately, nodes and arcs, respectively). Vertices are typically represented as points in the plane, and edges as curves connecting two vertices. Figure 1 shows three graphs relevant to our research. Some variations of graphs add more information to this basic structure, for instance, colored vertices or directed edges.

Some types of graphs that will be important in this paper are: complete graphs, (complete) bipartite graphs, k-partite graphs, planar graphs, directed graphs, paths, cycles, trees, and wheels. Complete definitions are available in Douglas B. West's *Introduction to Graph Theory* [22].

A **visibility graph** is a graph corresponding to a geometric representation[16, 21]. For instance, Figure 2 shows a **bar visibility graph** (BVG), where vertices u, v in a graph G are represented as a line segment (bar) parallel to the x-axis in the plane, and an edge is given between u and v if there is a vertical line segment between two bars that has endpoints on the bars corresponding to u and v, and does not cross any other bars. Note that collapsing each bar into a point provides a planar embedding of G [21].

Wismath (1985) proved that any planar graph G^+ is bar-visibility representable, where G^+ is the graph obtained by extending a graph G by one vertex v that is connected to all cutpoints of G. He further characterized the classes of graphs that admit bar-visibility representations by their associated cutpoints, also discussing weighted and directed cases. He left open the case of weighted, undirected graphs, claiming that such a question would require a partitioning problem [23].

Tamassia and Tollis (1986) extended several previous bar-visibility re-

Figure 2: A strong bar visibility graph with four vertices.

sults (including Wismath), establishing notation for several slightly different visibility rules:

1. **weak visibility**, based on an algorithm of Otten and van Wijk (1978), which allows certain edges in the bar-visibility construction to be ignored.

2. ϵ **visibility**, which expands visibility segments into visibility *bands* of nonzero width.

3. **strong visibility**, which extends weak visibility by including an "if and only if" clause, forcing *all* visibility bands to correspond to an edge.

They gave linear- or polynomial-time algorithms to construct all three types of representations from input graphs with the following conditions: (a) constructing a weak visibilty representation from any planar graph, (b) constructing an ϵ visibility representation from a 2-connected planar graph, and (c) constructing a strong visibility representation from a maximal planar graph and from a 4-connected planar graph [21].

Kant, Liotta, Tamassia, and Tollis (1993) proposed linear-time algorithms for creating both bar and rectangle visibility representations, for both rooted and free trees. They further discussed the area requirements of such diagrams, and proved that their algorithms construct diagrams with the minimal required area. They left open the questions of best possible constant factors, characterizing the class of graphs that admit rectangle visibility diagrams (with strong visibility rules), and characterizing the graphs with 2-ϵ representations (where no two visibility edges are allowed to cross) [16].

Fekete, Houle, and Whitesides (1995); Fekete and Henk (1999); and Bose et. al. (2002) study three-dimensional box visibility graphs [4, 12, 13].

Other variations on visibility rules are present in the literature, including bar **k-visibility**, where a visibility band can intersect up to k bars. Gethner and Laison (2011) showed the incomparability of unit bar $k-$visibility graphs and bar $k-$visibility graphs and discussed a family of $d-$box visibility graphs with nested K_8 rectangle visibility representations in their axis-aligned $2-$dimensional cross sections [14].

1.2 Rectangle Visibility Graphs

A **rectangle visibility graph** (RVG) is a visibility graph representable by rectangles with axis-parallel sides in the plane [5, 6, 7, 15, 20]. Two rectangles have an edge between their corresponding vertices in the graph if there is a vertical *or* horizontal band of sight between the sides of the two rectangles that does not intersect any other rectangles. Weak, ϵ, and strong variations, similarly to BVGs, have been studied. Figure 3 gives an example of a rectangle visibility graph, where the dashed red line shows a non-example of visibility.

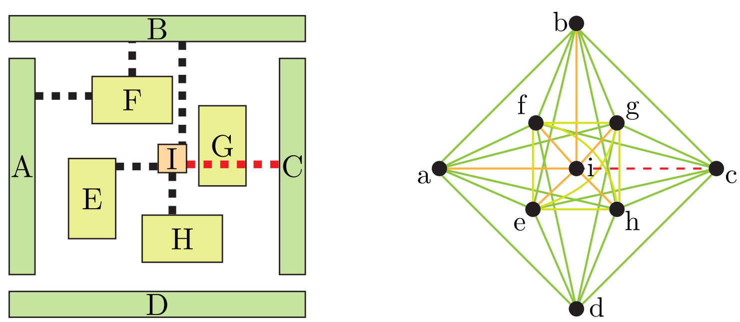


Figure 3: G_1, a maximal RVG on 9 vertices (34 edges).

Theorem 1 (Hutchinson, Shermer, Vince 1999 [15])**.** *A rectangle visibility graph on $n \geq 5$ vertices has at most $6n - 20$ edges.*

An RVG is at most thickness 2, a direct result following from the observation that an RVG is the union of two bar visibility graphs [15].

Bose, Dean, Hutchinson, and Shermer (1996) proved that certain classes of graphs are RVGs and gave efficient algorithms for constructing them.

Namely, for $1 \leq k \leq 4$, $k-$trees are RVGs. Graphs with caterpillar arboricity 2 are RVGs. Any graph whose vertices of degree four or more form a distance-two independent set is an RVG. And any graph with maximum degree 4 is an RVG [3].

Dean and Hutchinson (1997) discussed which bipartite graphs are RVGs. They showed that $K_{p,q}$ with $p \leq q$ is an RVG if and only if $p \leq 4$. They also showed that a bipartite RVG on $n \geq 4$ vertices has at most $4n - 12$ edges [6].

Streinu and Whitesides (2003) defined a **topological rectangle visibility graph,** which uses a pair of directed source-sink graphs to represent the horizontal and vertical relationships of a framed rectangle visibility diagram, cyclically ordered and allowing duplicate edges within this cyclic ordering (which the authors called multiplicity). They used such graphs to propose a quadratic-time algorithm which takes a pseudo-TRVG and constructs a rectangle diagram with minimum extent in each direction. They left open the question of how much information can be dropped from the TRVG and still keep the efficiency of their recognition algorithm [20].

1.3 Rectangle of Influence Graphs

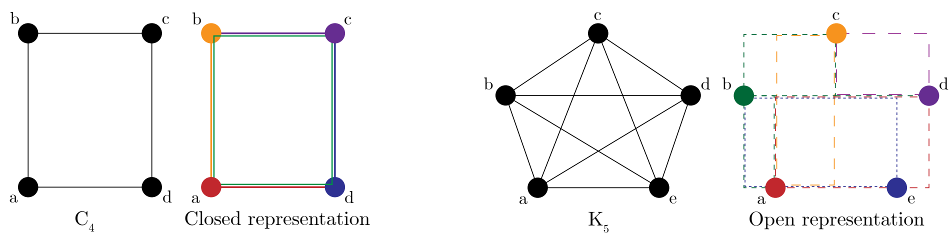
Let S be a set of points in the plane. Given two points p and q in S, the **rectangle of influence** between them is an axis-parallel rectangle such that p and q are on opposite corners of the rectangle of influence, as shown in Figure 4. When this rectangle of influence is empty, i.e. it contains no other points in S, we say p and q are **separated** (Alon et. al. 1985).

A (strong) **rectangle of influence graph (RIG)** is a visibility graph where vertices correspond to points in the plane. Vertices p and q are adjacent if and only if p and q are separated [1, 9, 10, 19, 24].

Weak RIGs are defined in previous literature, but will not be studied here (see [2, 8, 9] for more).

The boundary of the rectangle of influence is included in **closed** RIGs and not included in **open** RIGs (Liotta et. al. 1998). Figure 4 gives an example of a closed rectangle of influence drawing (4a) and an open rectangle of influence drawing (4b. Note that an open rectangle of influence is denoted with dashed lines, and a closed rectangle of influence with solid lines [18].

The closed rectangle of influence in Figure 4a *excludes* the edges ac and bd, since b and d are both on the boundary of the green rectangle of influence between a and c. If we used open rectangles of influence on this drawing, we would get those two additional edges, which would make this graph into K_4 as opposed to the cycle, C_4. Likewise, the open rectangles of influence in Figure 4b *include* edges ad and ed, since a closed rectangle of influence between a and e would contain both a and e, which would violate the emptiness condition.



(a) A closed RIG.

(b) An open RIG.

Figure 4: rectangle of influence representations of C_4 and K_5.

Theorem 2 (Alon, Füredi, Katchalski 1985 [1])**.** *A closed RIG on n vertices has at most $\left[\frac{n^2}{4}\right] + n - 2$ edges.*

The proof of this theorem relies on a theorem of de Bruijn, which applies a theorem of Erdös and Szekeres [11, 17]. The structure of this proof proved to be very useful in our research, as it is the basis of several of our own proofs involving edge bounds—Theorems 13 and 16. The following proof is adapted from Alon et. al. (1985), and has been edited for clarity [1].

Proof. The formula holds for RIGs on $2, 3,$ and 4 vertices since the number of edges in the complete graph is less than or equal to the bound.

Suppose $n \geq 5$. Let A be a set of n points in the plane. Let $s(A)$ be the number of pairs of points in A separated by empty rectangle of influences. Let $G(A)$ be the strong, closed rectangle of influence graph constructed from A.

Suppose by way of induction that the edge bound above holds for $n-2$, i.e. $s(A) \leq \left[\frac{(n-2)^2}{4}\right] + (n-2) - 2$.

Let a be a point of A whose $x-$coordinate is minimal. Let N be the set of points adjacent to a in $G(A)$. Let b be a point of N whose $x-$coordinate is maximal.

Note that any points x and y have an edge between their corresponding vertices in the graph if and only if there is no point z such that (x, z, y) is

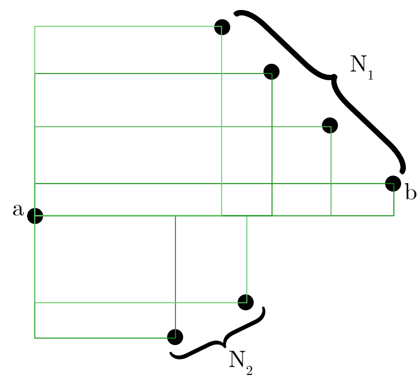


Figure 5: The definitions of a, b, N_1, and N_2.

weakly monotone, that is, their $y-$coordinates form a weakly increasing or weakly decreasing sequence.

Let N_1 be the points of N with y-coordinate greater than or equal to the y-coordinate of a, as in Figure 5. The points of N_1 cannot have the the same $x-$coordinates, or else one would be contained in the other's rectangle of influence with respect to a. Then if the points of N_1 are arranged in order of increasing $x-$coordinate, their $y-$coordinates must form a strictly decreasing sequence.

Let N_2 be the points of $N \backslash N_1$, that is all the points of N with $y-$coordinate less than the $y-$coordinate of a, again as shown in Figure 5 Similarly, the points of N_2 cannot have the same $x-$coordinates. If the points of N_2 are arranged in order of increasing $x-$coordinate, their $y-$coordinates must form a strictly increasing sequence.

Thus in $G(A)$, b is adjacent to at most two points of N. Then the number of edges in $G(A)$ incident to a or b is at most $(n-2)+2+1 = n+1$. Thus

$$s(A) \leq s(A \backslash \{a,b\}) + n + 1.$$

By the induction hypothesis,

$$s(A \backslash \{a,b\}) \leq \frac{(n-2)^2}{4} + (n-2) - 2.$$

Thus

$$s(A) \leq \frac{(n-2)^2}{4} + (n-2) - 2 + n + 1 = \frac{n^2}{4} + n - 2.$$

	Open RIGs	**Closed RIGs**
Representable	wheels paths $K_n, n \leq 8$	wheels cycles $K_n, n \leq 4$ trees with ≤ 4 leaves
Not Representable	non-path trees $K_n, n > 8$ cycles on > 3 vertices	trees with > 4 leaves $K_n,\ n > 4$

Table 1: rectangle of influence representability.

□

Table 1 summarizes findings of Liotta et. al.(1998) of what families of graphs are and are not representable using open and closed RIGs [18]. Figure 34a shows how these results relate to both RVGs and our own research.

2 Definitions and Preliminary Results

2.1 Corner Rectangle Visibility

Our main results concern a novel variation of the rectangle visibility problem, in which each rectangle has a single "eye" in one of its corners, rather than seeing from all four sides simultaneously. This variation turns out to be quite similar to the rectangle of influence idea, and our intuitive definition of corner rectangle sight can be further refined and formalized using rectangles of influence.

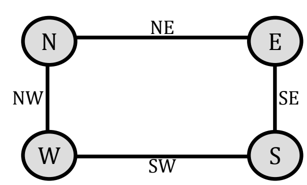


Figure 6: The four corners of a rectangle.

Formally, let S be a set of rectangles in the plane, with vertical and horizontal sides, whose interiors do not intersect. We label the four corners of a corner rectangle north, east, south, and west as shown in Figure 6. We denote the north corner of a rectangle A as A_N, the south as A_S, and so on for east and west. We further denote the $x-$coordinate of the (without loss of generality) north corner of A as $A_N(x)$, and its $y-$coordinate as $A_N(y)$. Furthermore, we label the sides of the rectangle NE, SE, SW, and NW to indicate which corners they are between. For each rectangle A in S, label a corner c_A of A as its **eye**, as shown in Figure 7. We call a rectangle with an designated eye a **corner rectangle**. A corner rectangle whose eye is at its north corner is a **north rectangle**. We similarly define **east rectangles**, **south rectangles**, and **west rectangles**. The **viewing region** of A is the unique closed quarter-plane R_A with corner c_A and intersecting A exactly at c_A. Suppose there is some rectangle B such that the intersection of B and the closed region R_A is nonempty. Let the **shadow** $\mathcal{U}_{A,B}$ of B with respect to A be the translated copy of R_A whose vertex is the corner of B exactly opposite the visibility direction of A, that is, if A is an east rectangle, the vertex of $\mathcal{U}_{A,B}$ would be the west corner of B.

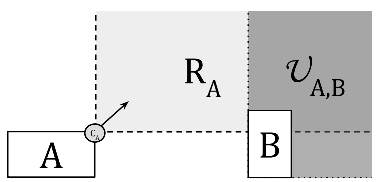


Figure 7: An example of c_A, R_A, and $\mathcal{U}_{A,B}$.

Now, with those definitions in hand, let us introduce the notion of sight for corner rectangles. A **sees** another rectangle B if:

1. B is not fully contained in $\mathcal{U}_{A,C}$ for some C intersecting R_A that is neither B nor A, and

2. the intersection of B with R_A has positive area.

We refer to these conditions as **sight condition 1** and **sight condition 2**. We can also define corner rectangle sight using rectangles of influence, as in the following lemma.

Lemma 3. *A sees B if and only if:*

1. *There exists a non-degenerate, closed rectangle of influence I from* c_A *to some point on the boundary of B,*

2. *I is tangent to A (i.e. I intersects A at exactly c_A), and*

3. *I does not contain any point of a rectangle C in S that is distinct from both A and B.*

We will refer to these as **lemma condition 1**, **lemma condition 2**, and **lemma condition 3**, respectively.

Proof. Let A, B, and C be distinct corner rectangles.

First, suppose that A sees B, that is, suppose sight condition 1 and sight condition 2. Notice that by sight condition 1, there must be some point b on the boundary of B in R_A that is not contained in $\mathcal{U}_{A,C}$. Now, consider the rectangle of influence I from the eye of A to b. We will show that I satisfies lemma conditions 1, 2, and 3.

1. By definition, I separates c_A and b, which is on the boundary of B. Furthermore, since $B \cap R_A$ has some positive area, b is not on the boundary of R_A. Therefore, I must be non-degenerate.

2. From sight condition 2, we know that the intersection of B with R_A is not empty. By construction, b is in R_A, and by definition the only point of A contained in R_A is c_A. Thus, I does not intersect any point of A other than c_A.

3. Since b is outside $\mathcal{U}_{A,C}$, the intersection of I with $\mathcal{U}_{A,C}$ must be empty. Recall that C is completely contained in $\mathcal{U}_{A,C}$ by definition—thus, the intersection of I with C must also be empty.

Therefore, all three conditions are satisfied when A sees B.

Second, suppose that lemma conditions 1, 2, and 3 hold for A, B, and C. We will show that sight conditions 1 and 2 follow from these assumptions.

Let b be an arbitrary point on the boundary of B and, as before, let I be the non-degenerate rectangle of influence between c_A and b (which follows from lemma condition 1).

1. By way of contradiction, suppose b is in the shadow $\mathcal{U}_{A,C}$. However, notice that since rectangles of influence and corner rectangles are axis-parallel, there must be at least one point in C with $x-$ or $y-$ coordinate equal to or less than that of b. However, this violates lemma condition 3. Having arrived at a contradiction, we conclude that if the arbitrary

b is in the shadow $\mathcal{U}_{A,C}$, B must be fully contained within $\mathcal{U}_{A,C}$; ergo, as long as there is some b which is *not* in $\mathcal{U}_{A,C}$, sight condition 1 will be satisfied.

2. It follows from lemma condition 2 and the definition of the visibility region R_A that I will be fully contained within R_A. Since b is included as a point on I, b must be within R_A. Since I is non-degenerate rectangle of influence by lemma condition 1, both the $x-$ and $y-$coordinates of b must be distinct from those of c_A (if either coordinate was equal, that would make I degenerate, i.e. a rectangle with no width, i.e. a line segment). Thus, b must not be exactly on the boundary of R_A, which implies that the intersection of B and R_A has some positive area.

Having satisfied sight conditions 1 and 2, we conclude that lemma conditions 1, 2, and 3 imply that A sees B.

We have shown both directions of the equivalence, therefore, we assert that the definition of sight by sight conditions 1 and 2 is equivalent to the definition given by lemma conditions 1, 2, and 3. □

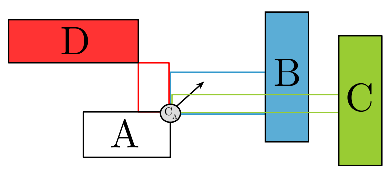
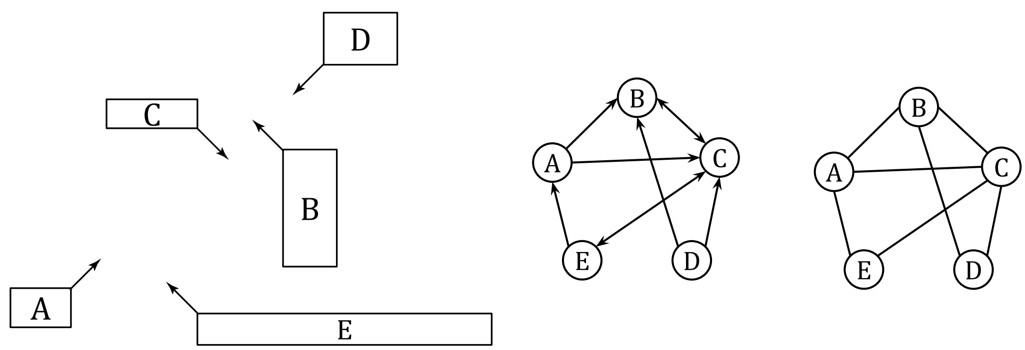


Figure 8: A definition of sight using rectangles of influence.

Figure 8 gives examples of rectangles of influence drawn from c_A to rectangles B, C, and D. The rectangle of influence between A and B has nonzero area, is tangent to A, and does not intersect any other rectangles. Further, we can see that $B \cap R_A$ is nonempty, and that B does not have any shadows cast upon it. Thus, under both definitions, A sees B. Notice that the rectangle of influence drawn between A and C contains points in B, which violates lemma condition 3. We can also observe that $C \cap R_A$ is fully contained within $\mathcal{U}_{A,B}$. Thus, C is not visible to A under either definition. Further, notice that the rectangle of influence between A and D intersects the northeast side of A, which violates lemma condition 2. This aligns with the observation that D does not intersect R_A.

2.2 Corner Rectangle Visibility Graphs

Given a set of rectangles S with viewing corners (Figure 9a), we construct a graph G with a vertex for each rectangle in S and an edge between two vertices a and b if and only if, for their corresponding rectangles A and B in S, either A sees B or B sees A (Figure 9c). We say that G is a **corner rectangle visibility graph** (CRVG) and that S is its **corner rectangle visibility representation**. We sometimes use directed edges (Figure 9b) to denote which rectangle is looking, and which is being looked *at*.



(a) Corner rectangle diagram. (b) Directed. (c) Undirected.

Figure 9: A CRVG representation and its CRVG with and without directed edges.

The **neighborhood** of a vertex a on a graph G, $N(a)$, is the subgraph composed of all the vertices adjacent to v and the edges between them [22]. Similarly, we define the **neighborhood** of a rectangle A in the CRV representation S, $N(A)$, to be the set of all rectangles that see A or are seen by A. Note that $N(A)$ is exactly the neighborhood of the vertex corresponding to A in G. This justifies our use of the same notation for neighborhoods in both the CRV representation and its corresponding CRVG. In general, we will use uppercase letters to label rectangles and the same lowercase letters to label their corresponding vertices in a graph. In a directed graph, the **in-neighborhood** $N_-(a)$ is the subgraph of $N(a)$ with edges directed towards a. Similarly, the **out-neighborhood** $N_+(a)$ is the subgraph of $N(a)$ where edges are directed away from a [22]. We use in- and out-neighborhoods to describe sight, so for a rectangle A, $N_+(A)$ is the set of rectangles seen by A and $N_-(A)$ is the set of rectangles that see A.

2.3 Monotonic Sequences

In our explorations of corner rectangle diagrams, two important types of monotonic sequences emerged: one where every rectangle in the sequence is visible from one direction, and one where every rectangle is visible from two directions.

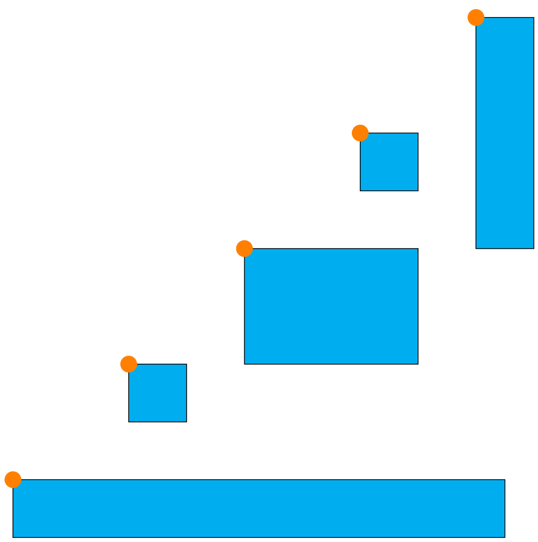


Figure 10: A sequence of N-monotone rectangles.

Let T be a set of rectangles and let D be one of the four cardinal directions: north, south, east, or west. The rectangles in T form a **D-monotone sequence** if for all rectangles T_i in T, the set of points $T_{i,D}$ (that is, the corner of T_i in direction D) forms a monotonically increasing or monotonically decreasing sequence. For example, for any A, B, C ordered by increasing $x-$coordinate in T, T is N-monotone if

$$A_N(y) < B_N(y) < C_N(y).$$

Proposition 4. *Any path can be drawn in a (south-facing) D-monotone sequence.*

Proof. An example of a D-monotone SCRVG representation of a path is given in Figure 11. More rectangles can be added in a similar manner to create a larger path. $\square$

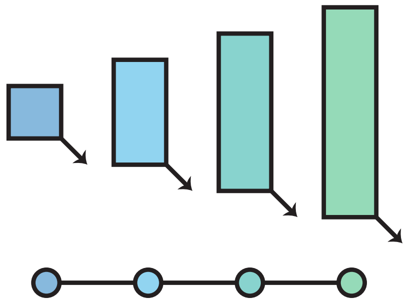


Figure 11: An N-monotone sequence of rectangles forming a path P_4.

The construction shown in Figure 11 is informally known as the “wifi construction” and is especially important in constructions later in the paper because it is both N-monotone and E-monotone.

Proposition 5. *Any tree can be drawn in a (south-facing) D-monotone sequence.*

Proof. A D-monotone SCRVG representation of an arbitrary tree is shown in figure 12. □

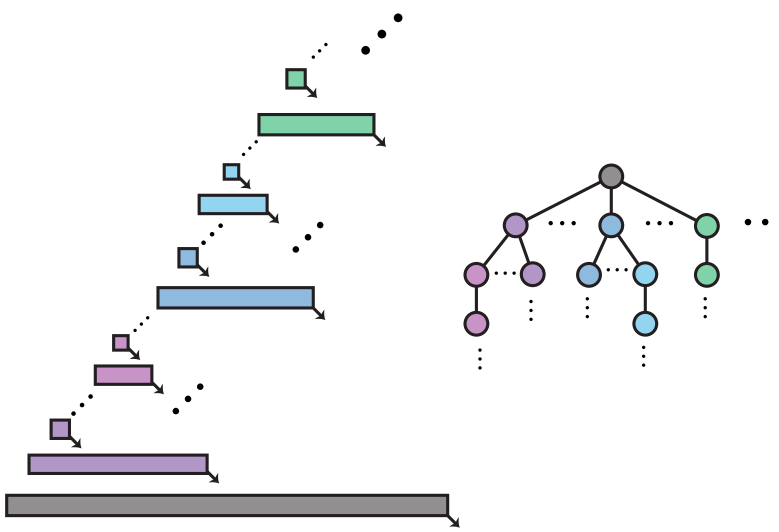


Figure 12: A D-monotone SCRVG representation of an arbitrary tree.

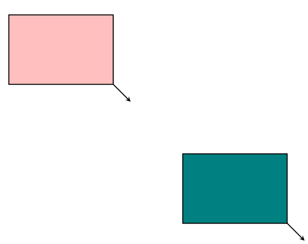
3 Results

3.1 South Corner Rectangle Visibility Graphs

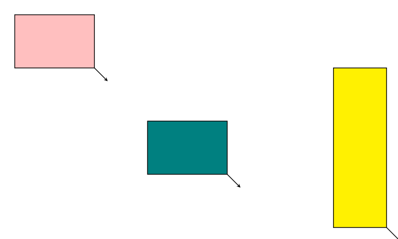
We now consider a variation of CRVGs which have only rectangles looking south. We call a CRVG whose representation only has south rectangles a **south corner rectangle visibility graph** (SCRVG). One reason SCRVGs are notable is that when considering directed graphs, they have no directed cycles, since no rectangle can look "up" or backwards, as all rectangles look "down." Also note that these results apply to any single-direction corner rectangle representation, e.g. where all rectangles look north, east, or west.

Proposition 6. *The complete graphs K_1, K_2, K_3, and K_4 are SCRVGs. The complete graph K_n is not an SCRVG for $n \geq 5$.*

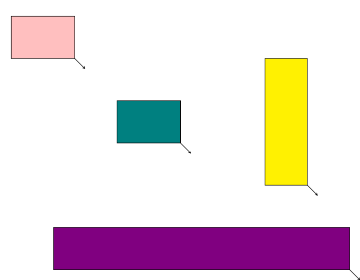
Proof. A single south rectangle is an SCRVG representation of K_1. SCRVG representations of K_2 K_2, K_3, and K_4 are shown in Figure 13. The fact that K_n is not an SCRVG for $n \geq 5$ follows from Theorem 13 below. $\square$



(a) An SCRVG representation of K_2.



(b) An SCRVG representation of K_3.



(c) An SCRVG representation of K_4.

Figure 13: Complete SCRVG Representations.

Proposition 7. *All complete bipartite graphs $K_{m,n}$ are SCRVGs for all positive integers m and n.*

Proof. An SCRVG representation of an arbitrary complete bipartite graph is shown in Figure 14. □

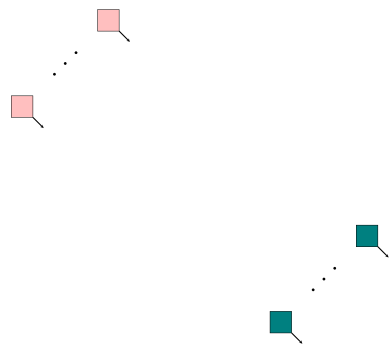


Figure 14: An SCRVG representation of an arbitrary complete bipartite graph.

Proposition 8. *All complete bipartite graphs with a path $K_{m,n} + P_n$ are SCRVGs for all positive integers m and n.*

Proof. An SCRVG representation of an arbitrary complete bipartite graph with a path is shown in Figure 15. □

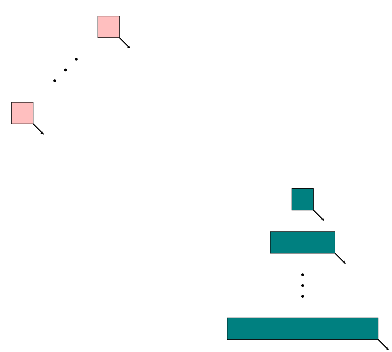


Figure 15: An SCRVG representation of an arbitrary complete bipartite graph with a path.

We will now prove some lemmas that we will use to prove Theorem 13, as well as other theorems later in the paper. For purposes of generality, we refer to an arbitrary direction as D and its opposite direction as $-D$. That is, if D is north then $-D$ is south, and similarly for the other directions.

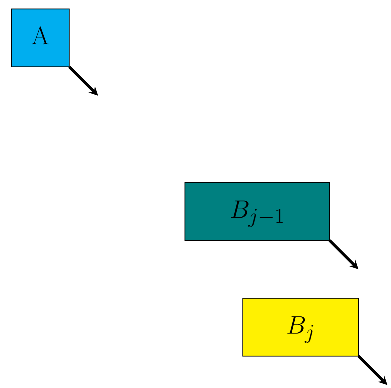


Figure 16: The south rectangle A cannot see B_j if the north corner of B_j has greater x-coordinate and smaller $y-$coordinate than the north corner of B_{j-1} in Lemma 9.

Lemma 9. *The out-neighborhood of a D-directional rectangle A in a CRVG representation is a $-D$-monotone sequence of rectangles.*

Proof. Without loss of generality, let A be a south rectangle. Let B_1, B_2, $\ldots$, B_n be the elements of $N(A)$, ordered by increasing $x-$coordinate of their north corners. Because B_i can be seen by A for all $1 \leq i \leq n$, $B_i \cap R_A$ must be a 2-D region in R_A that is not covered by a shadow.

By way of contradiction, suppose $B_1, \ldots, B_n$ do not form an N-monotone sequence. Thus, there is some B_j whose north $y-$coordinate is less than or equal to the north $y-$coordinate of B_{j-1}. However, notice that since the north corner of B_j has a greater $x-$coordinate and a smaller $y-$coordinate than the north corner of B_{j-1}, it is completely in the shadow $\mathcal{U}_{A,B_{j-1}}$, as in Figure 16. Thus we have a contradiction, since B_j cannot be visible to A if it is completely inside $\mathcal{U}_{A,B_{j-1}}$, but it was initially supposed that it *would* be visible to A. $\square$

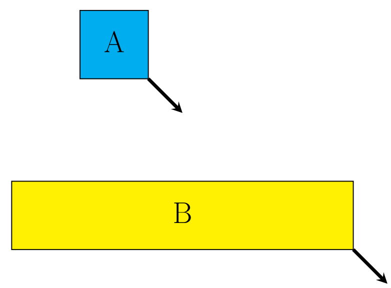


Figure 17: A rectangle B that is side-visible from A.

For a D-directional rectangle A, B is **side-visible** to A if A sees B but the $-D$ corner of B is not in R_A, as shown in Figure 17.

Lemma 10. *For a south rectangle A, there is at most one rectangle B side-visible from A where the sight is to the NW side of B, and similarly for other directions.*

Proof. Let A be a south rectangle. Let B be a rectangle side-visible from A where the sight is to the NW side of B. By way of contradiction, suppose there is another rectangle C, side-visible from A, where the sight from A is also to the NW side of C.

Because A is looking south at the NW sides of B and C, all $x-$coordinates of A are strictly less than the $x-$coordinates of B and C. Without loss of generality, suppose the $x-$coordinates of B are strictly less than the $x-$coordinates of C.

Then any rectangle of influence from the south corner of A to the NW side of C would intersect B. Thus C cannot also be side-visible to its NW side from A. □

Lemma 11. *For a rectangle A, there are at most two rectangles that are side-visible from A.*

Proof. Without loss of generality, let A be a south rectangle. Note that a rectangle that is side-visible from A must be side-visible to its NW or NE side. By Lemma 10, there is at most one rectangle side-visible from A to its NW side. By the same argument, there is at most one rectangle side-visible from A to its NE side. Thus there are at most two rectangles side-visible from A. See Figure 18 for an example. □

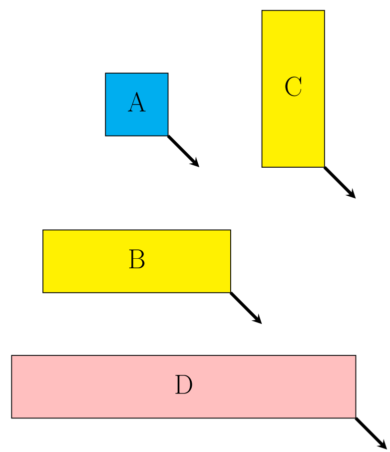


Figure 18: B and C are side-visible from A but D is not, as in Lemma 11.

Lemma 12. *A D-directional rectangle A in an $-D$-monotone sequence can see at most two other rectangles in the sequence.*

Proof. Without loss of generality, let X be an N-monotone sequence of rectangles and A be a south rectangle in X. By definition of N-monotone, the north corners of all rectangles in X are outside of R_A. Thus, any rectangles in X seen by A must be side-visible from A. By Lemma 11, there are at most two rectangles side-visible from A. Thus A can see at most two other rectangles in X. $\square$

A similar result is given by Corollary 21.

Theorem 13. *An SCRVG on n vertices has at most $\left[\frac{n^2}{4}\right] + n - 2$ edges, where $n \geq 2$, and this bound is tight.*

Proof. A representation of an SCRVG on n vertices with $\left[\frac{n^2}{4}\right] + n - 2$ edges is shown in Figure 19. We will now prove that an SCRVG on n vertices can have no more than $\left[\frac{n^2}{4}\right] + n - 2$ edges by induction.

The bound holds for 2, 3, and 4 vertices because the number of edges in the complete graph is less than or equal to the number of edges given by the formula.

Let $n \geq 5$, and let S be a set of n south rectangles and G be the graph represented by S. By way of induction, suppose that all SCRVGs on $n-2$ vertices have at most $\left[\frac{(n-2)^2}{4}\right] + (n-2) - 2$ edges.

Let A be the rectangle in S whose eye has minimal $x-$coordinate. Let $B_1, \ldots, B_k$ be the rectangles in $N(A)$. Since A is the rectangle with minimal $x-$coordinate, there can be no rectangles in S that see A. Thus $N(A) = N_+(A)$.

Since G is an SCRVG, it has no directed cycles. Thus, there must be some B_j that is not seen by any other rectangle in $N(A)$.

By Lemma 12, B_j can see at most two other rectangles in $N(A)$.

Then the number of edges incident to either a or b_j is $(n-2)+2+1 = n+1$ (that is $n-2$ edges between a and $b_i \neq b_j$, 2 edges from b_j to other vertices in $N(a)$, and 1 edge between a and b.)

Counting the edges in G gives $|E(G)| \leq E(G \backslash \{a, b_j\}) + n + 1$. By the induction hypothesis, $|E(G \backslash \{A, B_j\})| \leq \left[\frac{(n-2)^2}{4}\right] + (n-2) - 2$. Thus $|E(G)| \leq \left[\frac{(n-2)^2}{4}\right] + (n-2) - 2 + n + 1 = \frac{n^2}{4} + n - 2.$ $\square$

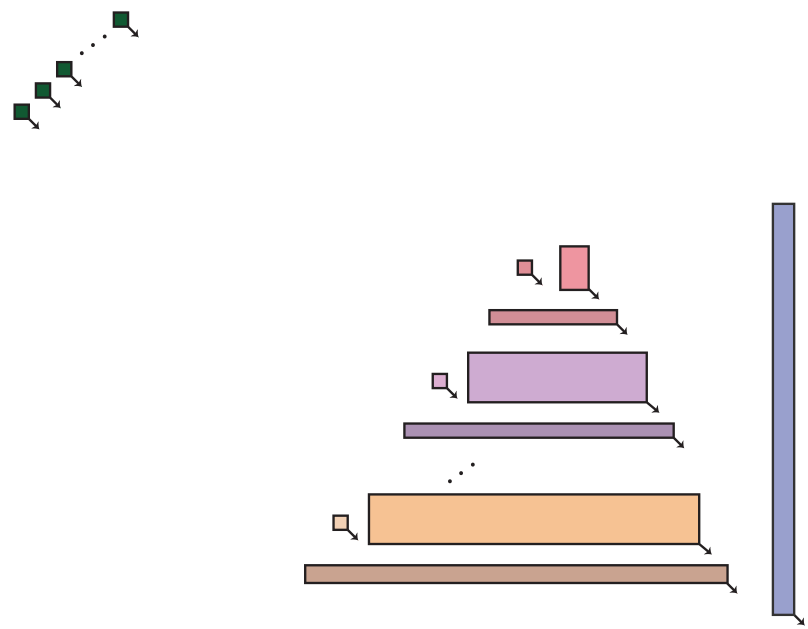


Figure 19: A representation of an SCRVG with n vertices and $\left[\frac{n^2}{4}\right] + n - 2$ edges.

3.2 South West Corner Rectangle Visibility Graphs

We will now study another variation of CRVGs which have only rectangles looking south or west and call them **south west corner rectangle visibility graphs** (SWCRVGs). When studying directed graphs, SWCRVGs also have no directed cycles, since similarly to SCRVGs, all rectangles look "down." The results on SWCRVGs also apply to any two-direction subset of CRVGs where the two directions are perpendicular.

Proposition 14. *All cycle graphs C_n are SWCRVGs for all positive integers $n \geq 3$.*

Proof. An SWCRVG representation of an arbitrary cycle is shown in Figure 20. □

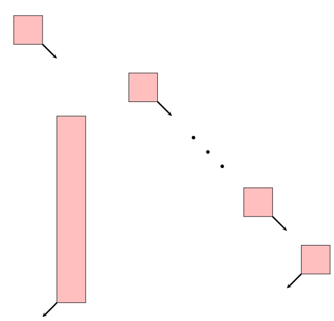


Figure 20: An SWCRVG representation of an arbitrary cycle.

We define $K_{j,l,m,n} + P_j$, a complete k-partite graph in which a partite set containing j vertices is connected by a path, for positive integers j, l, m, and n.

Proposition 15. *All complete 3-partite graphs with a path $K_{l,m,n} + P_n$ are SWCRVGs for all positive integers l, m, and n.*

Proof. An SWCRVG representation of an arbitrary 3-partite graph with a path is shown in Figure 21. □

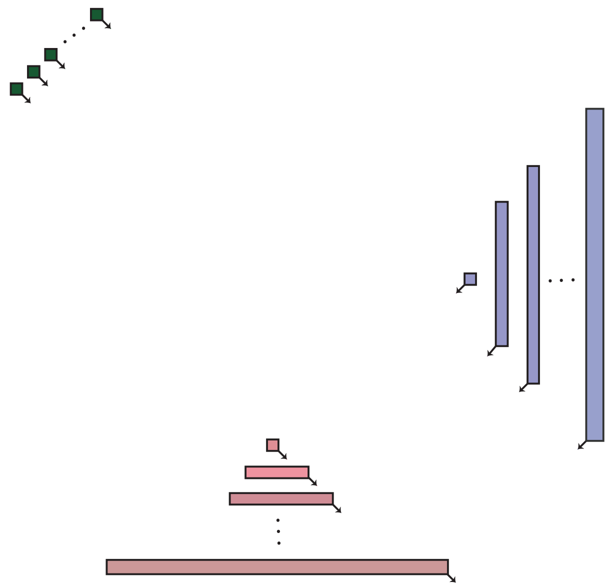


Figure 21: A representation of an SWCRVG with n vertices and $\left[\frac{n^2}{3}+\frac{n}{3}\right]-1$ edges.

Theorem 16. *An SWCRVG on n vertices has at most* $\left[\frac{n^2}{3}+\frac{n}{3}\right]-1$ *edges and the bound is tight.*

Proof. A representation of an SWCRVG on n vertices with $\left[\frac{n^2}{3}+\frac{n}{3}\right]-1$ edges is shown in figure 21. We will now prove that an SWCRVG on n vertices can have no more than $\left[\frac{n^2}{3}+\frac{n}{3}\right]-1$ edges.

The bound holds on 1, 2, 3, and 4 vertices because the number of edges in the complete graphs are less than or equal to the number of edges given by the formula. Thus we induct on $n \geq 5$.

Let S be a set of n south and west rectangles. Let G be the graph represented by S. By induction, suppose that all SWCRVGs on $n-3$ vertices have at most $\left[\frac{(n-3)^2}{3}+\frac{n-3}{3}\right]-1$ edges.

Note that there are no directed cycles in G. Let A be the rectangle in S whose eye has maximal $y-$coordinate. Thus there can be no rectangles that see A so $N(A) = N_+(A)$. Without loss of generality, suppose A is a south rectangle. Let B be a rectangle that is not seen by any other rectangles in $N(A)$. Let C be a rectangle that is not seen by any other rectangles in

$N(A) \cap N(B)$. Since B and C are seen by A, by Lemma 9, B and C are part of an N-monotone sequence.

Case 1. B **is a south rectangle**. Then the north corner of C is outside of R_B, but B sees C so C must be side-visible from B. Since $N(A)$ is an N-monotone sequence and B is a south rectangle, there are at most two rectangles in $N(A) \cap N(B)$. If C is one of those rectangles, there can be at most one rectangle in $N(A) \cap N(B) \cap N(C)$.

Case 2. B **is a west rectangle**. The visibility from B to C need not be side-visibility. If C is also a west rectangle, there is at most one rectangle in the N-monotone sequence of $N(A)$ that is also in $N(C)$. This rectangle may also be seen by B and thus be in $N(B)$. If C is a south rectangle, any rectangles in $N(A) \cap N(C)$ must be side-visible from C. By Lemma 11, there are at most two of these rectangles. However, at most one of them is in $N(B)$.

Then $|N(A) \cap N(B) \cap N(C)| \leq 1$. Thus the number of edges incident to a, b, and c is at most $(n-3)+(n-3)+1+3 = 2n-2$. That is, there are at most $n-3$ edges from each of a and b to vertices outside of $\{a, b, c\}$, at most 1 edge from c, and 3 edges between a, b, and c.

Thus $|E(G)| \leq E(G \backslash \{a, b, c\}) + 2n - 2$. By the induction hypothesis, $|E(G \backslash \{a, b, c\})| \leq \left[\frac{(n-3)^2}{3} + \frac{n-3}{3}\right] - 1$. Thus $|E(G)| \leq \left[\frac{(n-3)^2}{3} + \frac{n-3}{3}\right] - 1 + 2n - 2 = \left[\frac{n^2}{3} + \frac{n}{3}\right] - 1$. □

3.3 Corner Rectangle Visibility Graphs

In this section we provide CRVG representations for certain families of graphs.

Proposition 17. *All complete 4-partite graphs $K_{j,l,m,n}$ are CRVGs, for all positive integers j, l, m, and n.*

Proof. A CRVG construction of an arbitrary complete 4-partite graph is shown in Figure 22. □

With rectangles distributed as evenly as possible, a complete 4-partite graph on n vertices has $\lfloor \frac{3}{8}n^2 \rfloor$ edges. Note that by Theorem 13 and Theorem 16 a complete 4-partite graph is a CRVG that is neither an SCRVG nor an SWCRVG.

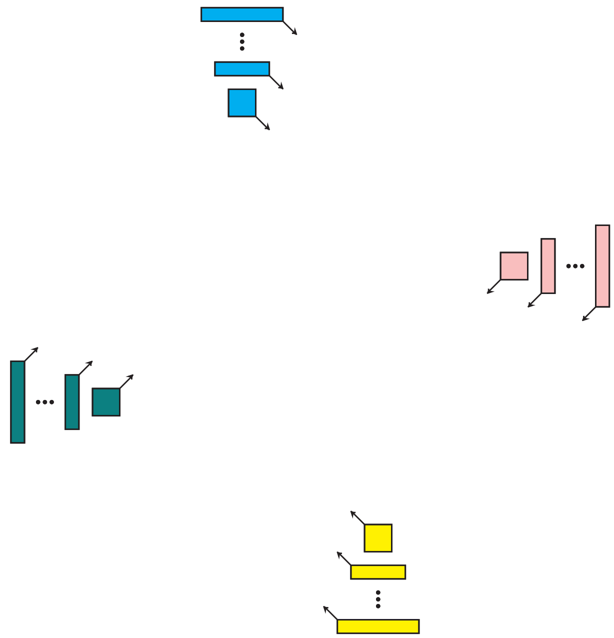


Figure 22: A CRVG representation of an arbitrary complete 4-partite graph.

Proposition 18. *All wheel graphs W_n are CRVGs.*

Proof. A CRVG representation of an arbitrary wheel graph is shown in Figure 23. □

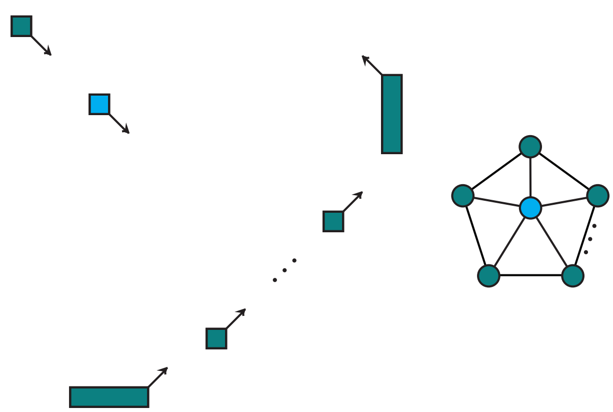
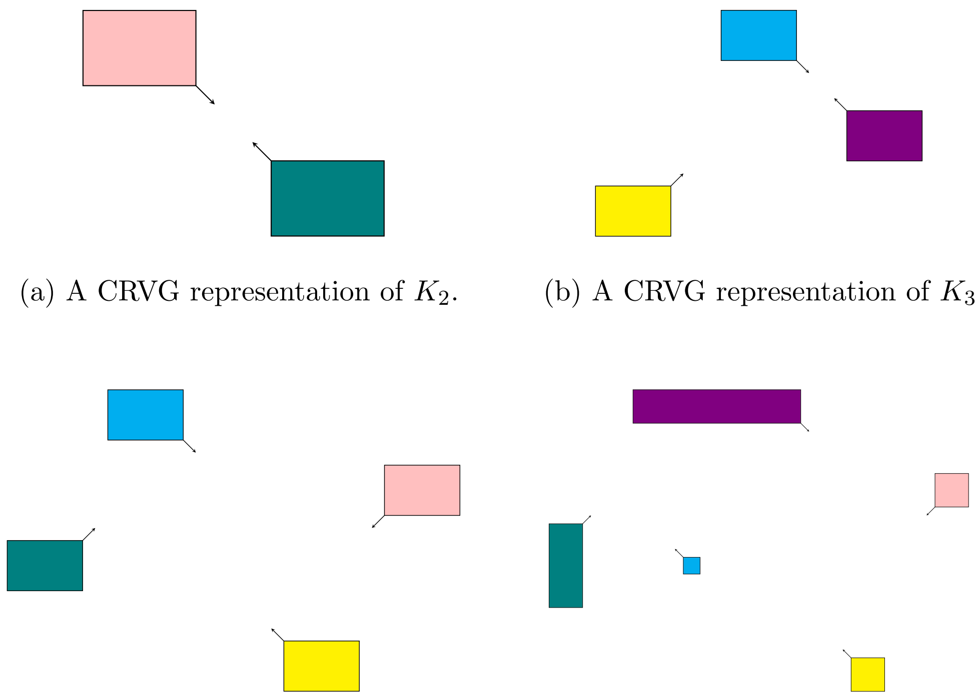


Figure 23: A CRVG representation of an arbitrary wheel.



(a) A CRVG representation of K_2. (b) A CRVG representation of K_3.

(c) A CRVG representation of K_4. (d) A CRVG representation of K_5.

Figure 24: Complete CRVG Representations.

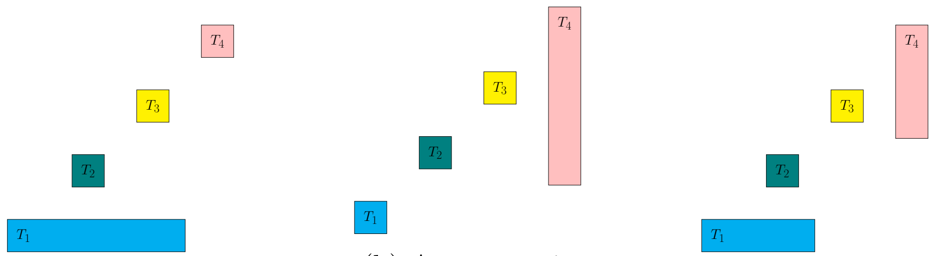
Proposition 19. *The complete graphs K_1, K_2, K_3, K_4, and K_5 are CRVGs. The complete graphs K_n for $n \geq 6$ are not CRVGs.*

Proof. Figure 24 shows CRVG representations for K_2 (24a), K_3 (24b), K_4 (24c), and K_5 (24a). Theorem 25 shows that K_n for $n \geq 6$ are *not* representable as CRVGs, i.e. K_5 is the largest complete CRVG. $\square$

Lemma 20. *In a CRVG representation of a directed graph, a rectangle in a clique of size at least 5 has out-degree at most 3.*

Proof. In a CRVG representation S, consider a corner rectangle A in a clique of size at least 5. By way of contradiction, suppose $N_+(A) > 3$. Without loss of generality, suppose A is a south rectangle.

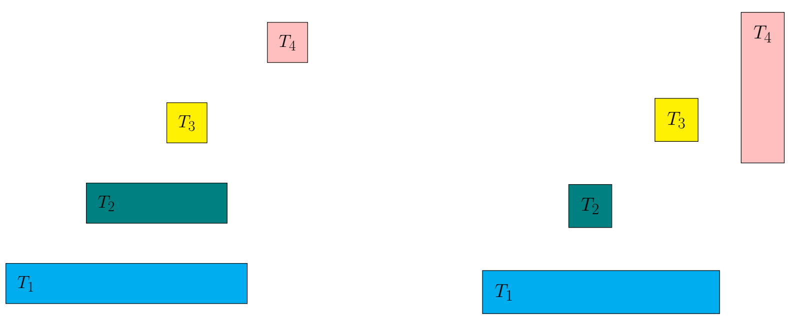
By Lemma 9, $N_+(A)$ is an N-monotone sequence of rectangles. Denote the rectangles in $N_+(A)$ as $N_+(A) = \{T_1, T_2, T_3, ..., T_k\}$ from left to right in S, with $k \geq 4$. Consider the rectangles T_1, T_2, T_3, and T_4. By our assumption, these rectangles form a clique of size 4. Therefore the rectangles T_1 and T_4 are adjacent in the graph. This can happen in one of three ways: either $T_{1,S}(x) > T_{3,S}(x)$, or $T_{4,W}(y) < T_{2,W}(y)$, or both $T_{1,S}(x) > T_{2,S}(x)$ and $T_{4,W}(y) < T_{3,W}(y)$, as shown in Figure 25. The first two cases are analogous.



(a) Case 1. (b) A symmetric representation of Case 1. (c) Case 2.

Figure 25: Three possible ways to create the $T_1 \sim T_4$ edge as described in Lemma 20.

Case 1. $T_{1,S}(x) > T_{3,S}(x)$**.** We also know that $T_2 \sim T_4$. This can happen in one of two ways: either $T_{2,S}(x) > T_{3,S}(x)$ or $T_{4,W}(y) < T_{3,W}(y)$ as shown in Figure 26.



(a) Case 1a. (b) Case 1b.

Figure 26: The subcases of Case 1 of Lemma 20.

Case 1a. $T_{2,S}(x) > T_{3,S}(x)$**.** In this case T_1 and T_3 can't see each other.

Case 1b. $T_{4,W}(y) < T_{3,W}(y)$**.** If T_4 sees T_3 then $T_{3,S}(y) < T_{4,S}(y)$ and $T_4 \not\sim T_1$. Therefore T_3 sees T_4 so T_3 is south or east. Therefore to get an edge between T_2 and T_3, either T_2 must face east and there is no edge between T_1 and T_2, or $T_{2,S}(x) > T_{3,S}(x)$ and $T_1 \not\sim T_3$.

Case 2. Both $T_{1,S}(x) > T_{2,S}(x)$ and $T_{4,W}(y) < T_{3,W}(y)$. This case encounters the same problem as **Case 1b.** That is, if T_4 sees T_3 then $T_{3,S}(y) < T_{4,S}(y)$ and $T_4 \not\sim T_1$. Therefore T_3 sees T_4 so T_3 is south or east. Therefore to get an edge between T_2 and T_3, either T_2 must face east and there is no edge between T_1 and T_2, or $T_{2,S}(x) > T_{3,S}(x)$ and $T_1 \not\sim T_3$.

Thus it is not possible for the N-monotone sequence of rectangles T_1, T_2, T_3, and T_4 to form a complete graph. Therefore, a corner rectangle A in a clique of size at least 5 has out-degree at most 3. $\square$

Corollary 21. *There is no representation of K_4 using four D-monotone rectangles, for any direction D.*

We will now prove some lemmas that will aid in the proof that K_6 is not a CRVG. We also introduce a type of diagram we will refer to as a dot diagram to help us describe properties of S, for example as shown in Figure 27. Each ellipse in the diagram represents the out-neighborhood of A, B, or C and is labeled with the name of the rectangle and its possible directions. Each dot represents a rectangle in S, labeled $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ from left to right in the diagram (so A, B, and C are three of these rectangles).

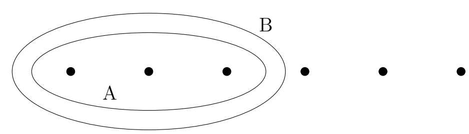


Figure 27: A dot diagram for Lemma 22.

Lemma 22. *Let S be a CRVG representation of K_6 and let A and B be rectangles in S with out-degree* 3. *Then $N_+(A) \neq N_+(B)$.*

Proof. If $N_+(A) = N_+(B)$ then A and B can't see each other, since neither A nor B can be in its own out-neighborhood. $\square$

Lemma 23. *Let S be a CRVG representation of K_6 and let A and B be rectangles in S with out-degree* 3. *If A and B are facing the same direction, then $N_+(A)$ and $N_+(B)$ must be disjoint.*

(a) $N_+(A) \cap N_+(B)$ contains 1 rectangle. (b) $N_+(A) \cap N_+(B)$ contains 2 rectangles.

Figure 28: Dot diagrams for Lemma 23.

Proof. Suppose A and B are facing the same direction and their out-neighborhoods are not disjoint. Without loss of generality, suppose they are both facing south. By Lemma 22, their out-neighborhoods intersect in one or two rectangles, as shown in Figure 28.

Since A and B are facing the same direction, in order to make an edge between A and B, one must be in the viewing region of the other. Say B is in the viewing region of A, as in Figure 29.

Consider the rectangle x_4. It is in $N_+(B)$ and not in $N_+(A)$. Therefore x_4 must see A. There is no way to place x_4 such that B sees x_4, x_4 sees A, and A doesn't see x_4 as shown in Figure 29.

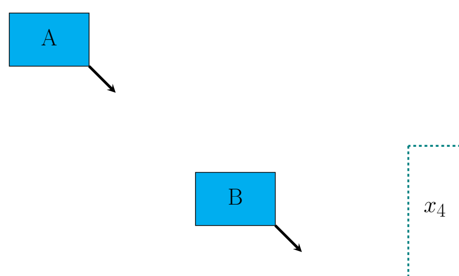


Figure 29: A CRVG representation of the dot diagram in Figure 28.

□

Lemma 24. *Let S be a CRVG representation of K_6 and let A and B be rectangles in S with out-degree 3. If A and B are facing opposite directions, then $N_+(A)$ and $N_+(B)$ must be disjoint.*

(a) $N_+(A) \cap N_+(B)$ contains 1 rectangle. (b) $N_+(A) \cap N_+(B)$ contains 2 rectangle.

Figure 30: Dot diagrams for Lemma 24.

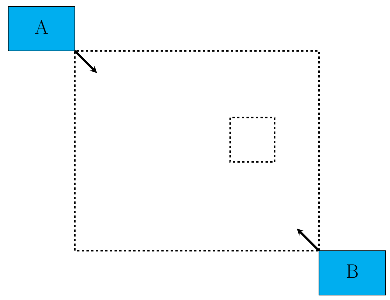


Figure 31: A CRVG representation for Lemma 24

Proof. Since A and B are adjacent, facing opposite directions, and both see an additional rectangle, their viewing regions must intersect. Thus A and B must be placed so they can see each other, as shown in Figure 31. However, no additional rectangles can be placed in the intersection of their viewing neighborhoods without blocking the sight between A and B. □

Theorem 25. *K_6 is not a CRVG.*

Proof. By way of contradiction, suppose S is a CRVG representation of K_6. By Lemma 20, a corner rectangle in a S has out-degree at most 3.

As there are 15 edges in K_6, the sum of the out-degrees of its vertices is at least 15. If it is exactly 15, the sequence of out-degrees of K_6 must be either $(3, 3, 3, 2, 2, 2)$, $(3, 3, 3, 3, 2, 1)$, or $(3, 3, 3, 3, 3, 0)$. Thus at least 3 rectangles in S must have out-degree 3. Call these 3 rectangles A,B, and C.

By Lemmas 22, 23, and 24, if the out-neighborhoods of two rectangles with out-degree 3 intersect, those rectangles must face in perpendicular directions. Without loss of generality, the out-neighborhoods and directions of A, B, and C are given by Figure 32.

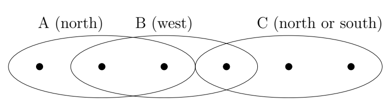


Figure 32: A dot diagram of with A, B, and C.

Since no rectangle can be in its own out-neighborhood and the out-neighborhoods of A and C cover all the rectangles, A must be in N_+C and C must be in $N_+(A)$. Thus the edge between A and C is bi-directed, and the total out-degree of the graph is at least 16. Therefore the sequence of out-degrees of K_6 must be either (3, 3, 3, 3, 2, 2), (3, 3, 3, 3, 3, 1), and a fourth rectangle D must have out-degree 3.

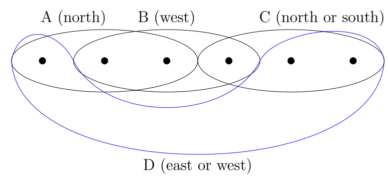


Figure 33: A dot diagram with A, B, C, and D.

Again by Lemmas 22, 23, and 24, D must face either east or west and its out-neighborhood is shown in Figure 33. By the same logic as before, B must be in $N_+(D)$ and D must be in $N+(B)$. Thus the edge between B and D is bi-directed, and the total out-degree of the graph is at least 17. Therefore the sequence of out-degrees of K_6 must be (3, 3, 3, 3, 3, 2), and a fifth rectangle E must have out-degree 3. There is no way to add the out-neighborhood of E to Figure 33 and we have a contradiction. □

Corollary 26. *CRVGs for $n = 7$ and $n = 8$ must be missing at least 2 and 3 edges from K_7 and K_8 (i.e. they have at most 19 and 25 edges, respectively).*

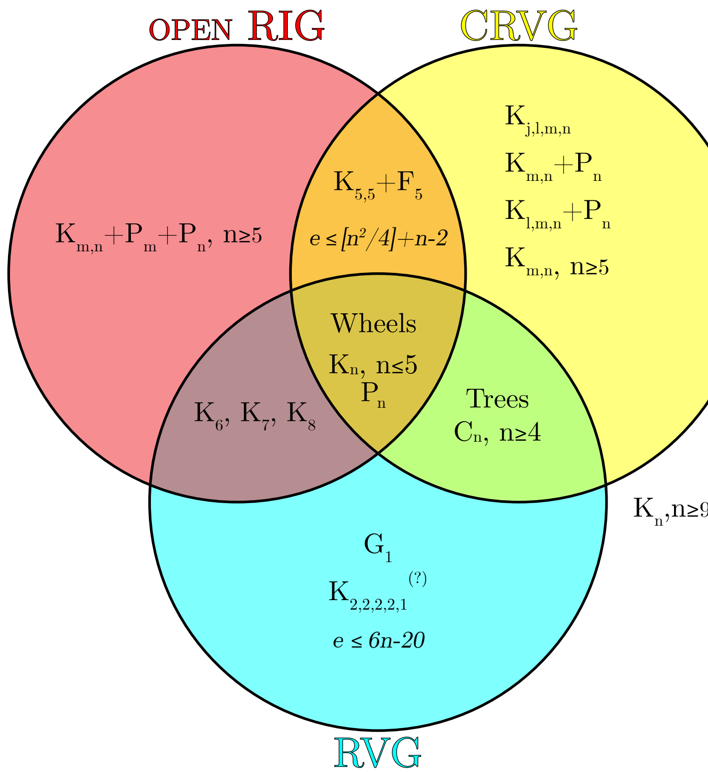
Proof. Any graphs on $n = 7$ vertices that have more than 19 edges contain an embedded K_6. Thus, by Theorem 25, such graphs are not possible.

A similar argument follows for $n = 8$ with $e > 25$. □

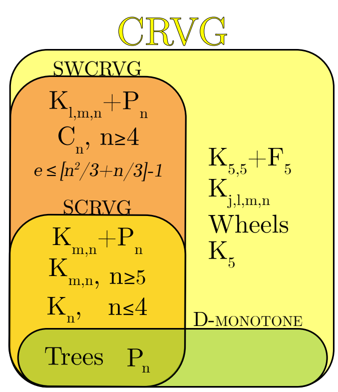
Note that we have examples of CRVGs with 7 vertices and 19 edges, and 8 vertices and 25 edges, so the bound in Corollary 26 is tight.

3.4 Classification Summary

The Venn diagram in Figure 34 shows various (families of) graphs that are rectangle visibility graphs [15], open rectangle of influence graphs [18], and corner rectangle visibility graphs. Figure 34b expands on the CRVG bubble in Figure 34a, showing which graphs are particular subcategories of CRVGs, namely D-Monotone CRVGs (Section 2.3), south CRVGs (Section 3.1), and south West CRVGs (Section 3.2).



(a) RIGs, CRVGs, and RVGs.



(b) Types of CRVGs.

Figure 34: Venn diagrams comparing graph representability with different rules.

4 Open Questions and Possible Extensions

There are many interesting questions that follow from our work. Some we have pondered more than others but all have much more potential for exploration.

1. Recall that all bar visibility graphs are planar [21, 23], but since K_5 is a CRVG, not all CRVGs are planar. Are all planar graphs CRVGs? Is there an algorithm to create a CRVG representation given a planar graph?

2. What is the maximum number of edges in a D-monotone CRVG?

3. What is the maximum number of edges in a CRVG? Since K_6 is not a CRVG but $K_6 - e$ is, we know the maximum number of edges in a CRVG with 6 vertices is 14. By Lemma 26, the first open case is $K_{2,2,2,2,1}$ with 9 vertices and 32 edges, referenced in Figure 34.

4. Can all SCRVGs with fewer edges than the SCRVG edge bound be drawn? What about the SWCRVG edge bound?

5. What other families of graphs are CRVGs?

6. Can CRVGs be recognized in polynomial time?

7. How would our results change if the definition of shadows cast by rectangles was changed to allow, for example, rectangles A and D to see each other in Figure 9?

8. What about corner rectangles which can see from more than one of their corners at once? Note that even if rectangles can see from all four of their corners, these graphs are distinct from RIGs since stars are representable this way but are not representable as RIGs in general.

5 Acknowledgements

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