Corner Rectangle Visibility Graphs

Juni DeYoung   Jayden Li   Lani Southern

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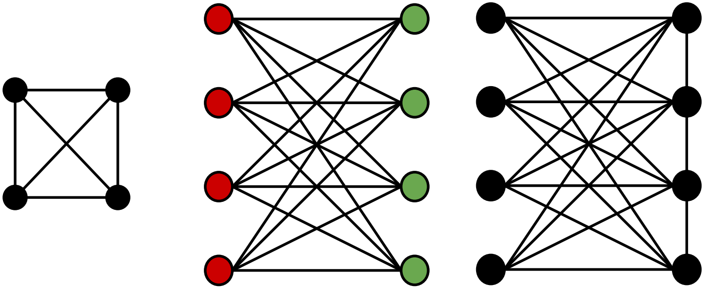
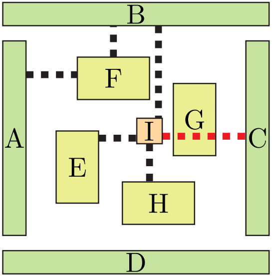
A graph is a finite set of vertices and a set of edges joining different pairs of distinct vertices.

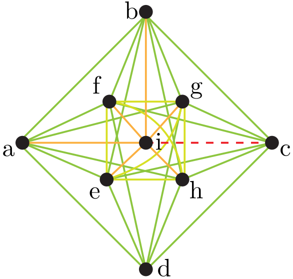


A complete graph (K_4), a complete bipartite $K_{4,4}$, and $K_{4,4}$ + path P_4

A rectangle visibility graph (RVG) is a visibility graph representable by rectangles with axis parallel sides drawn in a plane (Hutchinson, Shermer, Vince 1999) [2].

Two rectangles have an edge if there is a vertical or horizontal band of sight that does not intersect any other rectangles.

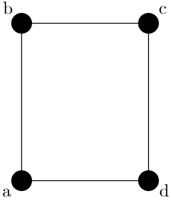




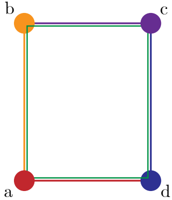
A maximal RVG for $n = 9$

A rectangle of influence graph (RIG) is a visibility graph with points in the plane. Edges are given if the rectangle of influence between two points is empty (Liotta et. al. 1998) [3].

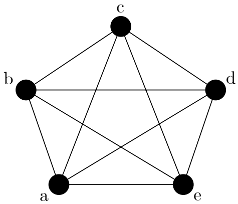
The boundary of the rectangle of influence is included in closed RIGs and not included in open RIGs.



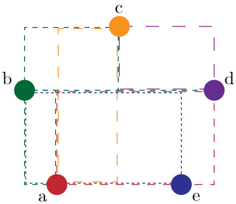
C_4



Closed representation



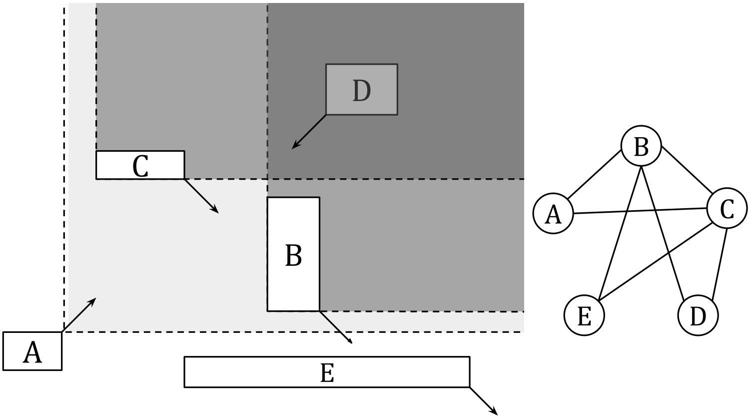
K_5



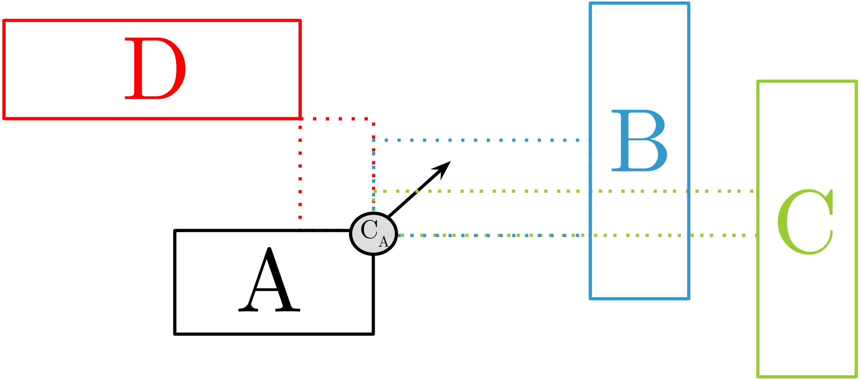
Open representation

CRVGs

A corner rectangle visibility graph (CRVG) is represented by rectangles in the plane. The rectangles "see" out of one corner, and other rectangles can cast shadows that block sight.

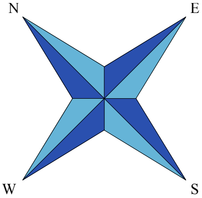


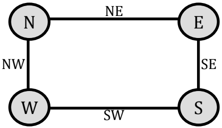
A rectangle *A* can "see" another rectangle if the open rectangle of influence to the other rectangle in the direction of the arrow contains no other rectangles.

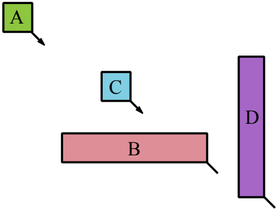


CRVGs

We use cardinal directions to specify which way a rectangle is "looking."



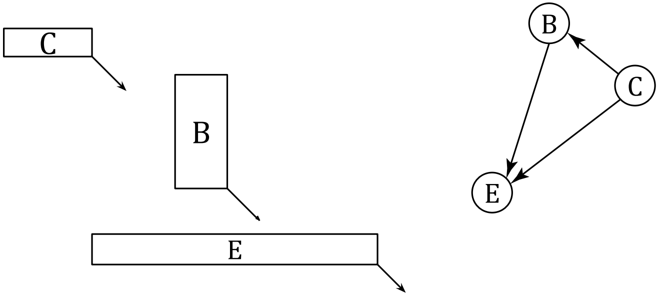




south rectangle rep of K_4

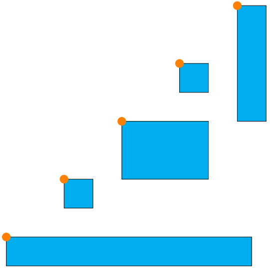
We will also study subsets of CRVGs with special restrictions, for example, all the rectangles look south.

A south-facing corner rectangle visibility graph (SCRVG) is a CRVG where every rectangle and its associated visibility region look south.



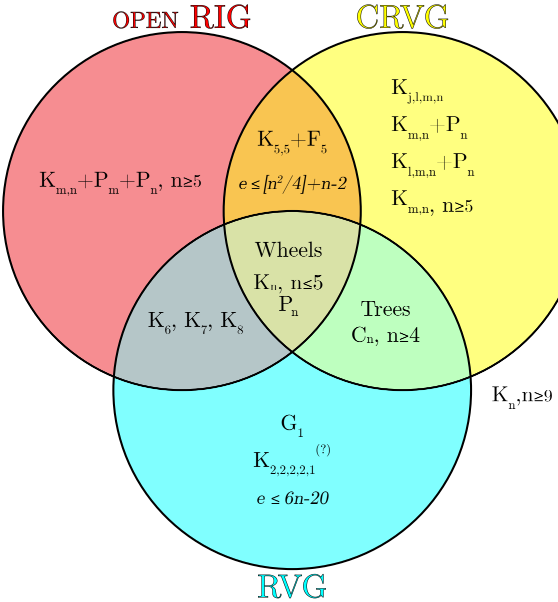
A south-facing subset of the diagram on slide 5.

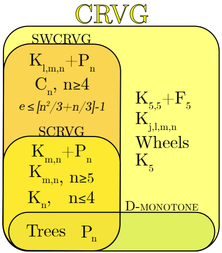
An *N*-monotone CRVG has the north corners of its rectangles as shown below.



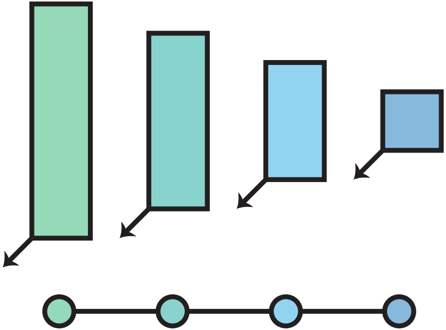
An *N*-monotone sequence of 5 rectangles.

Examples of CRVG, RIG, and RVG



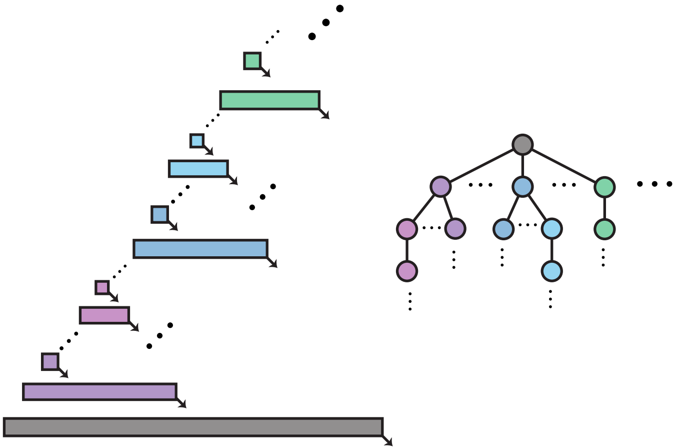


CRVG Classification: Paths



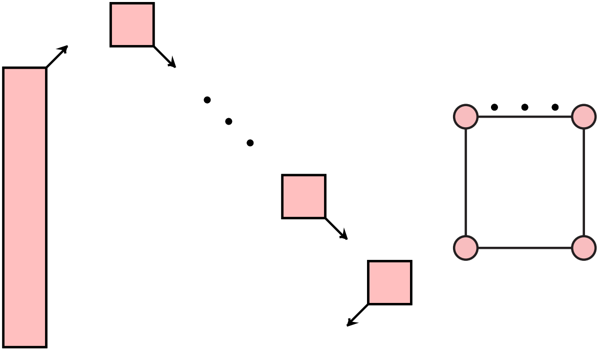
A D-monotone CRVG representation of a path P_n, aka the “wi-fi" construction.

CRVG Classification: Trees



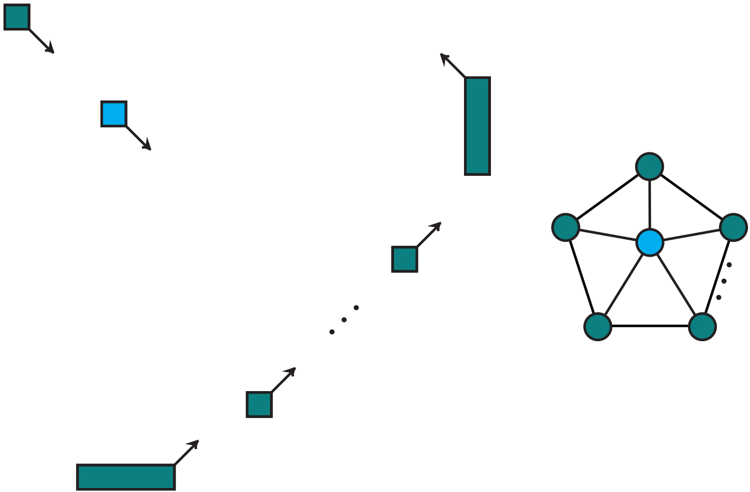
A *D*-monotone SCRVG representation of a tree.

CRVG Classification: Cycles



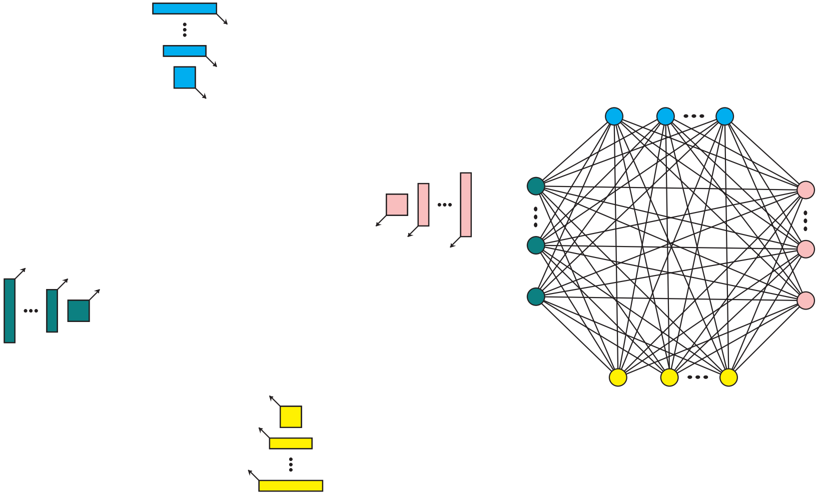
A CRVG representation of a cycle C_n.

CRVG Classification: Wheels



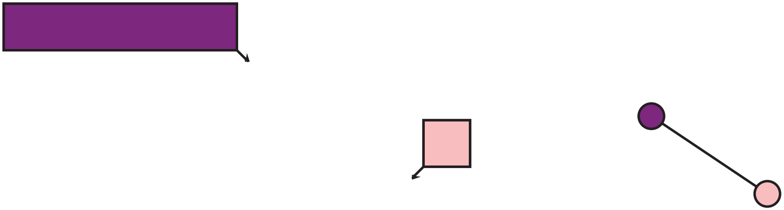
A CRVG representation of a wheel W_n.

CRVG Classification: Complete k-partite Graphs



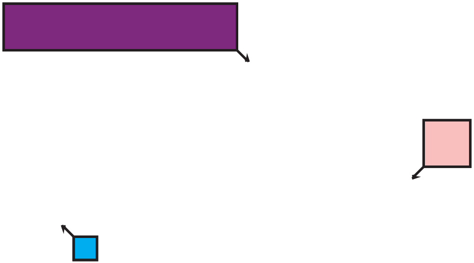
A CRVG representation of a complete 4-partite graph.

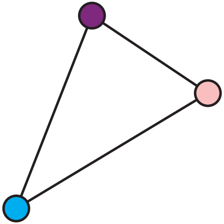
CRVG Classification: Complete Graphs



A CRVG representation of K_2

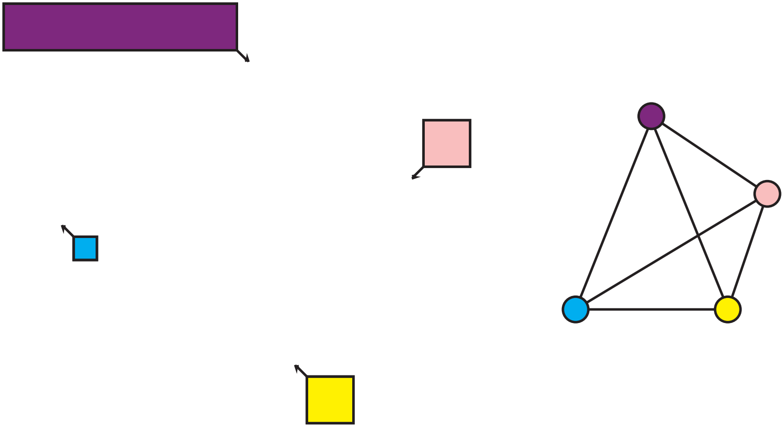
CRVG Classification: Complete Graphs





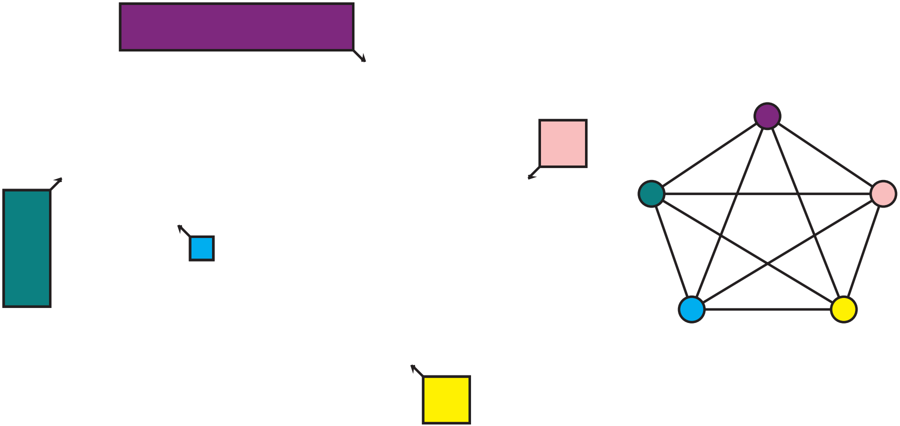
A CRVG representation of K_3

CRVG Classification: Complete Graphs



A CRVG representation of K_4

CRVG Classification: Complete Graphs

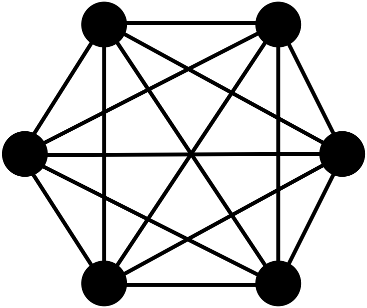


A CRVG representation of K_5

No K_6

Theorem

K_6 is **not** a CRVG.

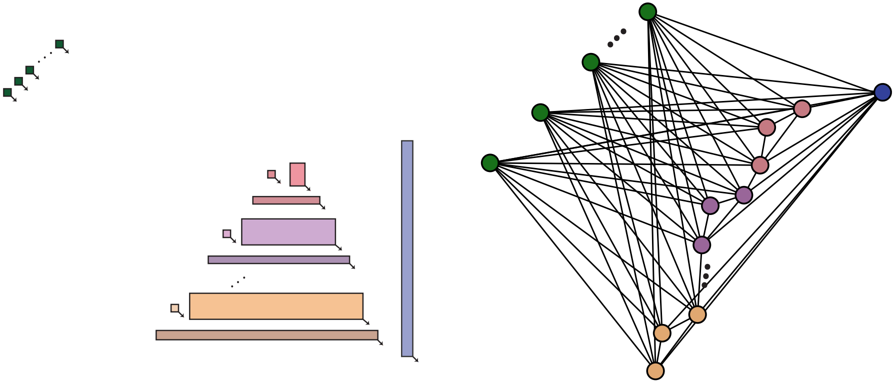


A question of interest

What is the maximum possible number of edges for a visibility graph on n vertices?

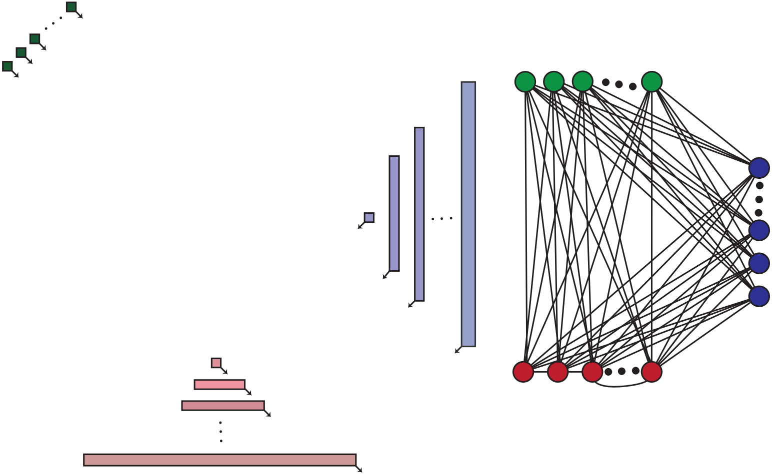
- Closed RIGs: $[\frac{n^2}{4}] + n - 2$ edges (Alon et. al. 1985)[1]
- RVGs: $6n - 20$ edges (Hutchinson, Shermer, Vince 1999)[2]
- SCRVGs: $[\frac{n^2}{4}] + n - 2$ edges
- SWCRVGs: $[\frac{n^2}{3} + \frac{n}{3}] - 1$ edges
- D-monotone CRVGs: $[\frac{n^2}{4} + \frac{n}{2}] - 1$ edges (?)
- General CRVGs: Open Question

Edge Bound: SCRVGs



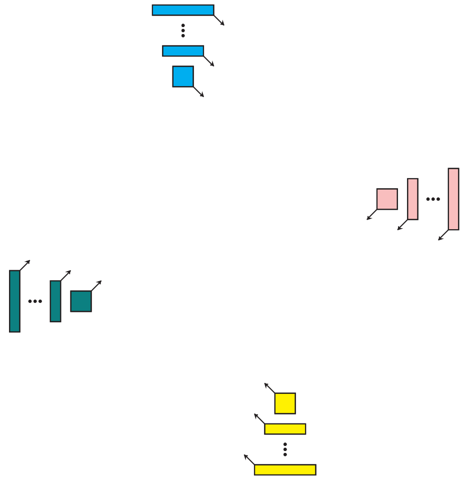
An SCRVG on n vertices has at most $[\frac{n^2}{4}] + n - 2$ edges.

Edge Bound: SWCRVGs



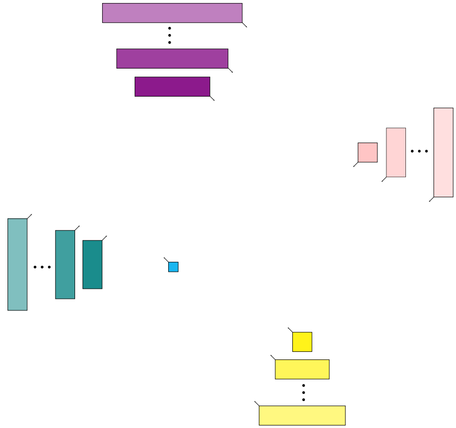
An SWCRVG on n vertices has at most $[\frac{n^2}{3} + \frac{n}{3}] - 1$ edges.

Edge Bound Attempts: CRVGs



$K_{j,l,m,n}$ has $\lfloor \frac{3}{8} n^2 \rfloor$ edges.

Edge Bound Attempts: CRVGs



K_5 plus Wi-Fi spurs has $\left\lceil \frac{3n^2-2n+15}{8} \right\rceil$ edges.

Corner Rectangle Visibility Graphs, Part II

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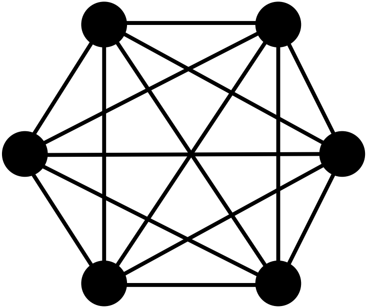
April 19, 2023

No K_6

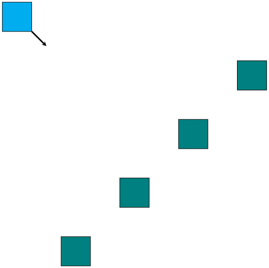
Theorem

K_6 is **not** a CRVG.



Max out degree 3

The rectangles that can be seen by a *D* directional rectangle form a monotone sequence in the opposite direction.

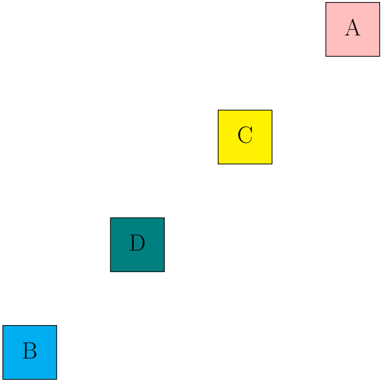


Lemma

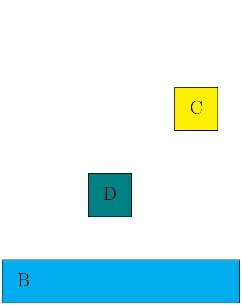
The biggest complete graph made by a *D*-monotone sequence of rectangles is K_3.

The out degree of a rectangle is the number of rectangles it can see.

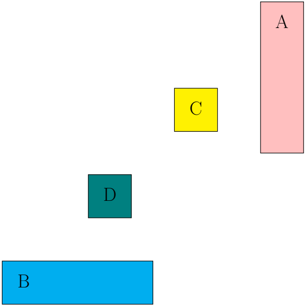
We will prove this by attempting to form a K_4 from a monotone sequence of four rectangles and arrive at a contradiction.
Fix the north corners of four rectangles in a monotone sequence.



Step 1: Get *AB* edge.
Case 1: *B* extends to the right of *C*.
Case 2: *B* extends to the right of *D* and *A* extends below *C*.



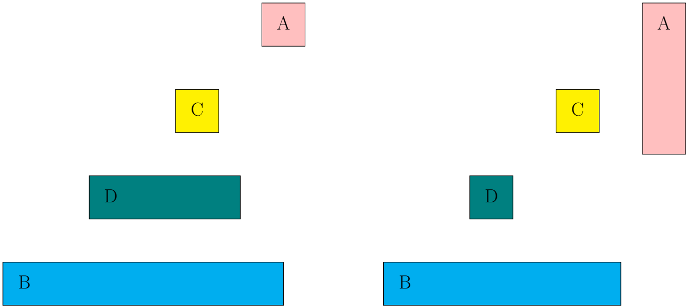
Case 1



Case 2

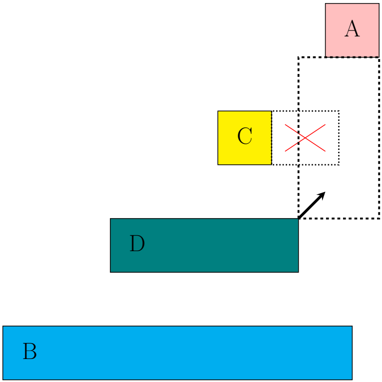
Max out degree 3

Case 1: *B* extends to the right of *C*.
Step 2: Get *AD* edge.
Case 1a: *D* extends to the right of *C*.
Case 1b: *A* extends below *C*.

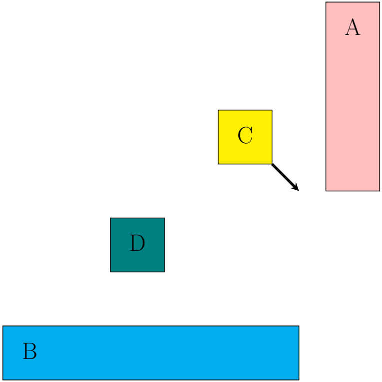


Case 1a   Case 1b

Case 1a: *D* extends to the right of *C*.
We cannot get the *BC* edge.

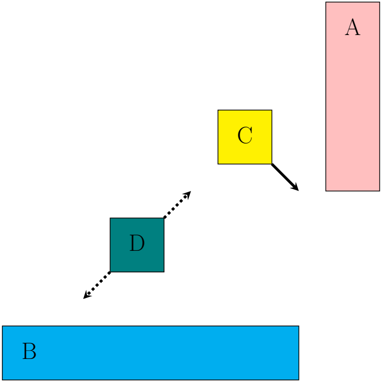


Case 1b: *A* extends below *C*.
Step 3: Get *AC* edge.

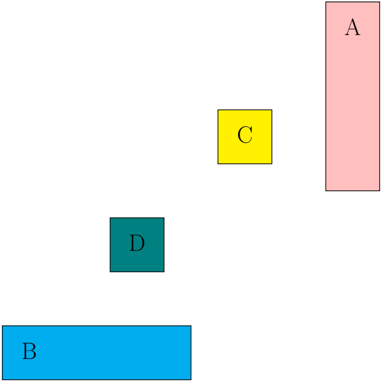


C must look at *A*.

We can only get one of *CD* and *BD* edges.



Case 2: B extends to the right of D and A extends below C.



See **Case 1b**.

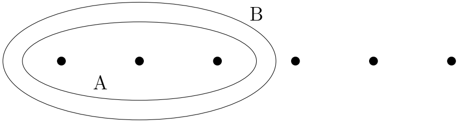
No K_6

Theorem

K_6 is not a CRVG.

K_6 has out-degree of 15 $\Rightarrow$ maximum of $3(6) = 18$ out-degree $\Rightarrow$ we have to subtract 3 extra $\Rightarrow$ only 2 possible cases: out degree $(3, 3, 3, 1, 1, 1)$ and out degree $(3, 3, 3, 3, 2, 1)$.
$\Rightarrow$ At least 3 rectangles in K_6 guaranteed to have out-degree of 3 (name them A,B,C).

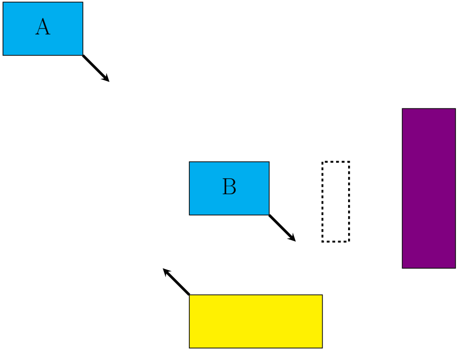
The out-neighborhoods of A, B cannot be equal, no matter what directions A, B are facing.



Since no rectangle can be its own out-neighborhood, the out-neighborhood of A and B must be other rectangles.
None of these first 3 dots are A or B, A and B cannot see each other.

No K_6

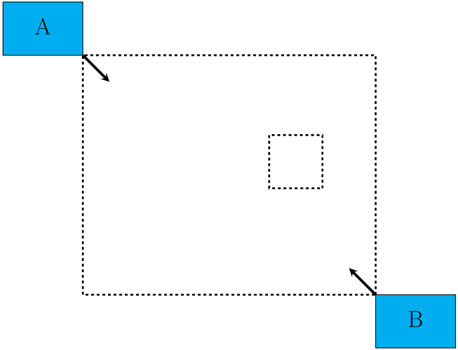
If A, B facing the same direction,



The rectangle with dashed sides cannot extend past the bottom of B.

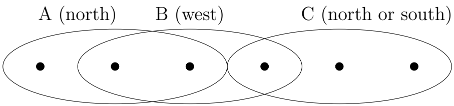
No K_6

If A, B facing opposite direction,



Any rectangles that both A and B can see will cast shadows to one another, thus their out-neighborhoods must be disjoint.

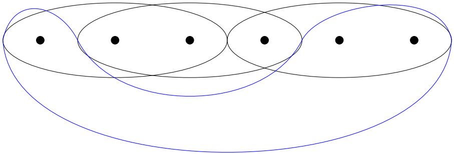
No K_6



No rectangle can be in its own out-neighborhood $\Rightarrow$ *A* and *C* sees each other, $A \leftrightarrow C$, they are bi-directed.
$\Rightarrow$ to get the right number of edges, a fourth rectangle *D* with out-degree 3 must exist.

No K_6

This is the only way to draw the out-neighborhood of D:



Again, rectangles cannot be in their own out-neighborhoods $\Rightarrow$ B and D sees each other, $B \leftrightarrow D$, they are bi-directed.
Thus, a fifth rectangle with out-degree 3 must exist, but it cannot be added to the diagram. ■

Open Questions

- What is the maximum number of edges in a CRVG?
- What other families of graphs are or are not CRVGs? SCRVGs? SWCRVGs?
- For CRVGs that are also RVGs, can we find an algorithm to find a CRVG representation from an RVG representation?

Thanks and Questions

Special thanks to Professor Josh Laison, our research advisor.
Thanks to August Bergquist, Ezekiel Jakob Druker, and Cin Vhetin for contributions in Junior Research Seminar, Spring 2022.

[1] Noga Alon, Z. Füredi, and M. Katchalski.
Separating pairs of points by standard boxes.
European J. Combin., 6(3):205–210, 1985.

[2] Joan P Hutchinson, Thomas Shermer, and Andrew Vince.
On representations of some thickness-two graphs.
Computational Geometry, 13(3):161–171, 1999.

[3] Giuseppe Liotta, Anna Lubiw, Henk Meijer, and Sue H Whitesides.
The rectangle of influence drawability problem.
Computational Geometry, 10(1):1–22, 1998.